Lecture 11

In this lecture, we'll start a detailed study of the symmetric group S_n. We learned about S_3 in lecture 3 and now we will study S_n, $\forall\ n \geq 3$. First, lets recall

Definition Let B be a non-empty set. A permutation of B is a function from B to B which is a bijection, i.e., it is both one to one and onto.

When B is a finite set then we can take $B = \{1, 2, \ldots, n\}$ if B has n elements. The group S_n is defined as follows:-

Definition Let $B = \{1, 2, \ldots, n\}$. The set of all permutations of B is the group S_n under the operation of composition of functions. It is called the symmetric group of degree n.

Suppose $\alpha \in S_n$, then $\alpha : \{1, \ldots, n\} \to \{1, 2, \ldots, n\}$ and α is a bijection. We saw in Lec. 3 about another way to represent α,

$$\alpha = \begin{bmatrix} 1 & 2 & \cdots & n \\ \alpha(1) & \alpha(2) & \cdots & \alpha(n) \end{bmatrix}$$

Exercise Prove that S_n is a group and $|S_n| = n! = n \cdot (n-1) \cdot \cdots 1$.

Exercise Prove that $\forall\, n \geq 3$, S_n is non-abelian.

Hint:- Construct two functions in such a way that they give different functions by changing the order of composition.

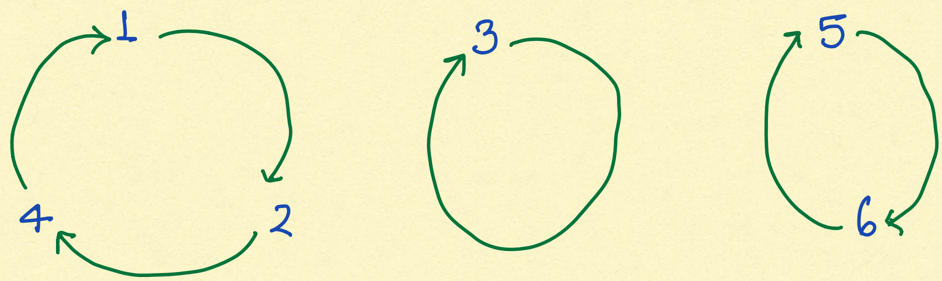
Cycle Notation

Since S_n has $n!$ elements, if we keep representing them as a matrix, it'll become cumbersome. Also, it's hard to deduce new properties about S_n with that notation. The cycle notation is very advantageous in that respect.

We'll understand cycle notation through examples. Let $\alpha \in S_6$ given by

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 6 & 5 \end{bmatrix}$$

This can be schematically represented as follows



i.e., we are starting with 1, draw a cycle to the element $\alpha(1)$ (2 in this case). The draw next cycle to $\alpha(\alpha(1))$ (in this case $\alpha(2)=4$). Keep drawing it till we reach 1. We know we'll reach 1 eventually as α is onto.

Once a cycle is done, start with the element not yet represented in the cycle, which is 3 in this case. Since $\alpha(3) = 3$ so the cycle starts from 3 and ends at 3.

Restart from the elements left in both the cycles, i.e. 5 and 6 and repeat the procedure.

However we can't keep drawing arrows to represent cycle, so we simply write

$$\alpha = (1\,2\,4)(3)(5,6)$$

which is telling us that α

* maps 1 to 2, 2 to 4 and 4 again to 1, i.e. $(1 \dashrightarrow 2 \dashrightarrow 4)$.
* maps 3 to 3 hence (3).
* maps 5 to 6 and 6 to 5, hence $(5\,6)$.

In fact, (3) is just telling us that $\alpha(3) = 3$, so we'll omit that too and just write

$$\alpha = (1\,2\,4)(5\,6)$$

cycle notation for α

Keeping in mind that since $\alpha \in S_6$ and 3 is missing from the notation, so $\alpha(3) = 3$.

Let's see another example. Let $\beta \in S_8$ with

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 5 & 7 & 6 & 1 & 4 & 8 \end{bmatrix}$$

We start with 1. Since $1 \to 3$, we start with $(1\,3\ldots$. Now $3 \to 5$, so we write $(1\,3\,5\ldots)$. $5 \to 6$ so, $(1\,3\,5\,6\ldots)$. Finally $6 \to 1$ so the cycle is complete, hence $(1\,3\,5\,6)$.

Now the first element not represented in the above is 2 and $2 \to 2$, so we leave it. The next element not represented is 4 and $4 \to 7$, so we write $(4\,7\ldots)$. And $7 \to 4$ so the cycle is completed and we write $(4\,7)$.

Finally $8 \to 8 \Rightarrow$ we leave it. So combining above, i.e. in the cycle notation, we see

$$\beta = (1\,3\,5\,6)(4\,7)$$

Observe that above representation is giving us all the informations about β and is much simpler to write.

Let's see one more example before we see the uses of the cycle notation.

Let $\gamma \in S_5$ and

$\gamma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 3 & 1 \end{bmatrix}$. Following the procedure above, we see

$$\gamma = (1\,2\,4\,3\,5)$$

Note that the identity element in S_n maps every element to itself so how do we write that in the cycle notation? Well, if ϵ is the identity element of, say S_4, then we simply write any of the element of $\{1,2,3,4\}$ in a bracket. So,

$$\epsilon = (1) \text{ or } \epsilon = (2) \text{ or } \epsilon = (3) \text{ or } \epsilon = (4).$$

<u>Remark</u> :- Just remember that any element missing in the cycle notation means that it is mapped to itself.

<u>Question</u> How to see the group operation in the cycle notation?

Suppose $\alpha, \beta \in S_7$ and $\alpha = (1\,2\,4)(3\,5\,6)$ and $\beta = (1\,3\,5\,4\,6)$. What is $\alpha \cdot \beta$? First of all remember that when we compose functions then we first apply β and then α. Now the process is simple: $\alpha.\beta \in S_7$, so

* Start with 1. $\alpha.\beta(1) = \alpha \circ \beta(1) = \alpha(\beta(1))$.

Now $\beta(1) = 3$ and $\alpha(3) = 5 \Rightarrow \alpha.\beta(1) = 5$

so we get $(1\,5\ldots)$.

* Next we take 5. $\alpha.\beta(5) = \alpha(\beta(5)) = \alpha(4) = 1$

So $\alpha.\beta(5) = 1$ and the cycle stops, so we get $(1\,5)$.

* The first element not represented above is 2. So $\alpha.\beta(2) = \alpha(\beta(2)) = \alpha(2) = 4$, because we haven't written 2 in $\beta \Rightarrow \beta(2) = 2$. So $\alpha.\beta(2) = 4$ hence we get $(2\,4\ldots)$.

* We look at $\alpha.\beta(4) = \alpha(\beta(4)) = \alpha(6) = 3$.

So we get $(2\,4\,3\ldots)$.

* We look at $\alpha.\beta(3) = \alpha(\beta(3)) = \alpha(5) = 6$. So we get $(2\,4\,3\,6\ldots)$.

* We look at $\alpha.\beta(6) = \alpha(\beta(6)) = \alpha(1) = 2$. Hence the cycle completes and we get $(2\,4\,3\,6)$.

* Finally the only element left is 7.

$\alpha.\beta(7) = \alpha(\beta(7)) = \alpha(7) = 7$. So we won't write it. So we get

$$\alpha.\beta = (1\,5)(2\,4\,3\,6).$$

We end with two definitions.

<u>Definition</u> An expression of the form $(a_1, a_2, \cdots a_m)$ is calle a cycle of length m or an m-cycle.

<u>Definition</u> A 2-cycle is called a transposition.

<u>Exercise</u> Practise the cycle notation with arbitrary permutations of your choice.

o ———— x ———— x ———— o